GRASP Laboratory
General Robotics & Autonomous Systems and Processes

Mechatronic Modeling and Design with Applications in Robotics

Analytical Modeling (Part 2)

Transfer Function

t: time.

$s = \sigma + j\omega$ — σ: Real, jω: imaginary, ω: frequency.

- A system is assumed to be at rest (zero initial conditions),
- A transfer function is defined by

$$G(s) := \frac{C(s)}{R(s)}.$$

Laplace transform of **system output**

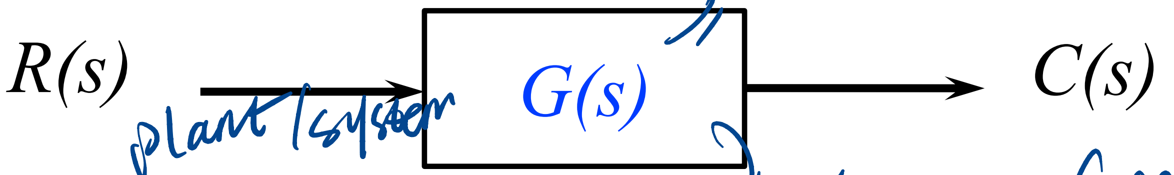
Laplace transform of **system input**

$s = \sigma + j\omega$. frequency domain

$\frac{\text{output}}{\text{input}} \rightarrow$ Transfer Function (TF)

$G(s)$

input



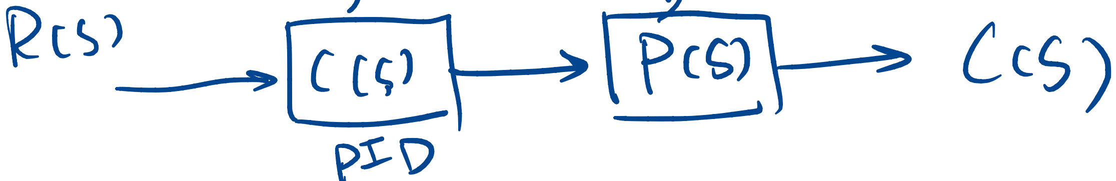
plant/system

$\frac{C(s)}{R(s)}$.

output.

$G(s) = C(s)\,P(s)$

controller



Transfer Function

Analytical Model in frequency domain

LT $t \to s$

$\mathcal{L}^{-1}\{LT\}$ $s \to t$.

Note: input, system and output into three separate and distinct parts.

A general *n*th-order, linear, time-invariant differential equation:

$$\mathcal{L}\left\{a_n \frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0 r(t)\right\}$$

C(s) → output R(s) → input

$TF = \dfrac{\text{output}}{\text{Input}}$.

$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$

where *c(t)* is the output, *r(t)* is the input.

Assume: zero initial conditions, and take the Laplace transform on both side:

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)C(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)R(s)$$

output output input.

$$TF = \frac{C(s)}{R(s)} = \frac{(b_m s^m + a_{m-1}s^{m-1} + \cdots + a_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}$$

output input

$$TF = G(s) = \frac{C(s)}{R(s)}$$

$$C(s) = R(s)G(s)$$

Example

s ~~time~~

Find the transfer function represented by $\frac{dc(t)}{dt} + 2c(t) = r(t)$, and use the result to find the response $c(t)$ to a unit step input with zero initial conditions.

$\frac{dc(t)}{dt} + 2c(t) = r(t) \rightarrow$ Laplace transform

$sC(s) + 2C(s) = R(s) \rightarrow$ TF: $\frac{\text{output}}{\text{input}} = \frac{C(s)}{R(s)} = \frac{1}{s+2}$.

input

$R(s) \rightarrow \boxed{G(s)} \rightarrow C(s)$ output

system model (TF)

$C(s) = R(s) \cdot G(s) = \frac{1}{s} \cdot \frac{1}{s+2} = \frac{1}{s(s+2)}$

output $\mathcal{L}^{-1}\{C(s)\}$.

? time domain

partial fraction expansion

Laplace Transform Table

$\frac{1}{s(s+2)} = \frac{1}{s} \times \frac{1}{(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+2}$

$= \frac{K_1(s+2) + K_2 s}{s(s+2)}$

K_1? K_2?

$\begin{cases} (K_1 + K_2) \cdot s + 2K_1 = 1 \\ K_1 + K_2 = 0 \end{cases} \Rightarrow \begin{matrix} K_1 = \frac{1}{2} \\ K_2 = -\frac{1}{2} \end{matrix}$

$\frac{1}{s(s+2)} = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s+2}$.

Laplace Transform

One of the most important math tool in the course!

system equation

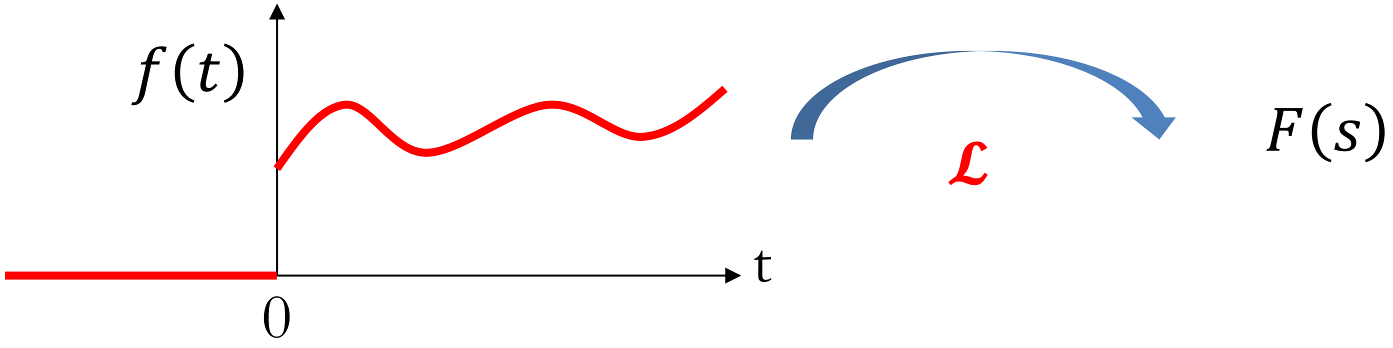
Definition:

For a function $f(t)$ $(f(t) = 0\ for\ t = 0)$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

(s: complex variable)

$F(s)$ is denoted as the Laplace transform of $f(t)$



Inverse Laplace Transform

Allow us to find $f(t)$ given $F(s)$:

s domain

a input

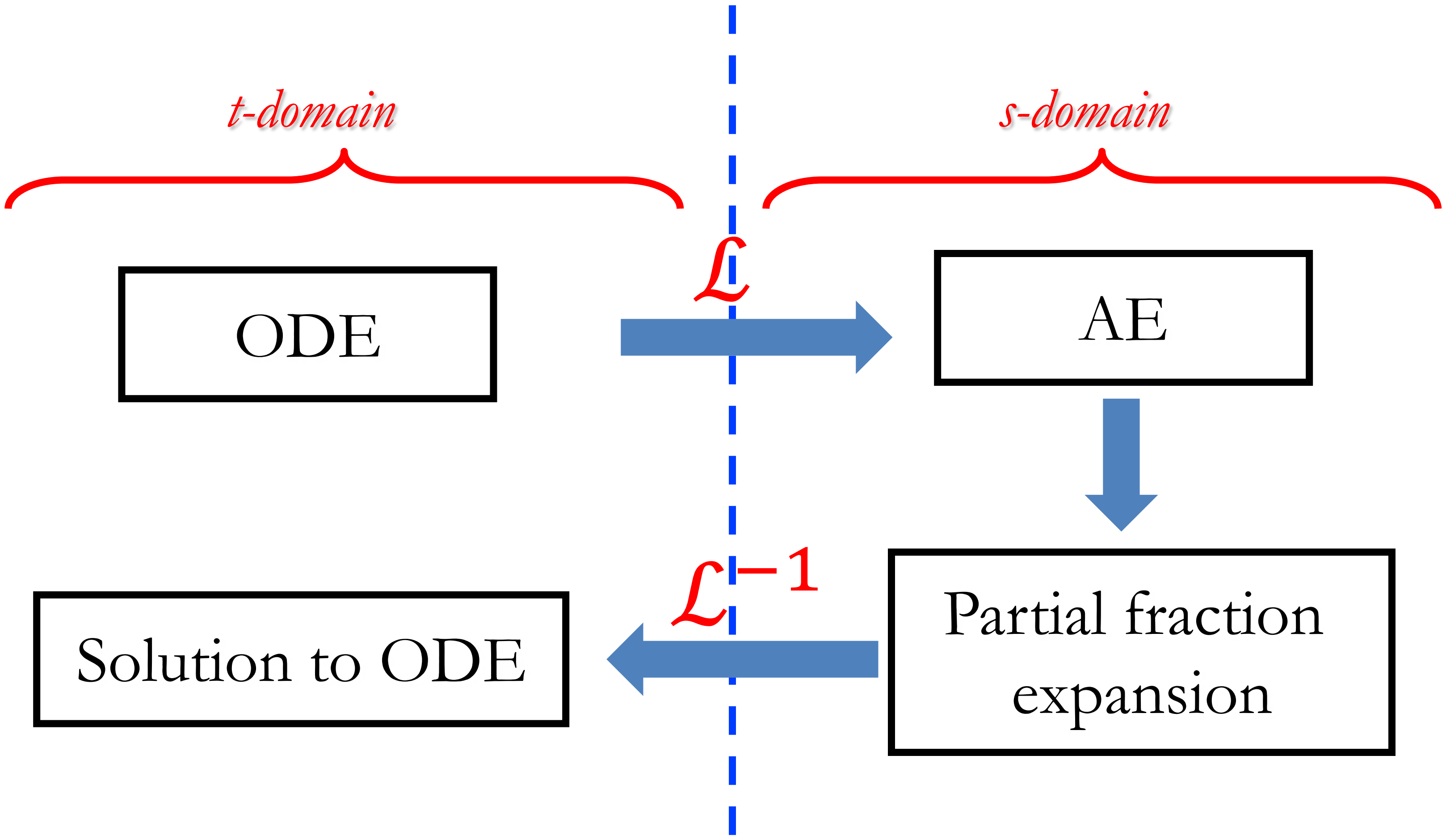
$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds = f(t)u(t)$$

where

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

An Advantage of Laplace Transform

Transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Laplace Transform Table

No.	$f(t)$	$F(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{-at}u(t)$	$\frac{1}{s+a}$
6	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

$\xrightarrow{\mathcal{L}}$

$\xleftarrow{\mathcal{L}^{-1}}$

Laplace Transform Theorems (Properties)

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}\left[e^{-at} f(t)\right] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT} F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0-) - f(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

Partial-Fraction Expansion

Partial-Fraction Expansion: To convert the function to a sum of simpler terms.

E.g.,

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

Partial-Fraction Expansion

$$\Rightarrow \quad F(s) = s + 1 + \underline{\frac{2}{s^2 + s + 5}}$$

Reminder:
Order of the numerator less than its denominator

$\mathcal{L}^{-1}$

$$\Rightarrow \quad f(t) = \mathcal{L}^{-1}\{s\} + \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2 + s + 5}\right\}$$

3 Cases (Roots of the Denominator)

1. Real and Distinct

$$F(s) = \frac{2}{(s+1)(s+2)} \rightarrow F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

K_1 ? K_2

$$\frac{K_1(s+2) + K_2(s+1)}{(s+1)(s+2)}$$

$K_1(s+2) + K_2(s+1) = 2.$

$K_1(s+2) + K_2(s+1) = 2.$

$K_1 s + K_1 2 + K_2 s + K_2 = 2.$

$$\begin{cases} K_1 + K_2 = 0. \\ 2K_1 + K_2 = 2. \end{cases}$$

K_1 K_2?

2. Real and Repeated

$$F(s) = \frac{2}{(s+1)(s+2)^2} \rightarrow F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

K_1 K_2 K_3

3. Complex or Imaginary

$$F(s) = \frac{3}{s(s^2+2s+5)} \rightarrow F(s) = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

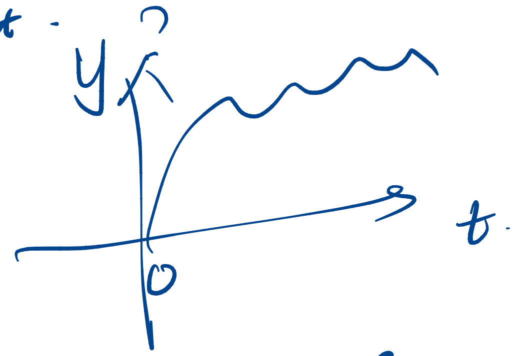
K_1 K_2 K_3?

Differentiation Theorem: $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$; $\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0) - f'(0)$;
$\mathcal{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s) - \sum_{k-1}^{n} s^{n-k}\ f^{k-1}(0)$;

Example: Given the following differential equation, solve for $y(t)$ if all initial conditions are zeros.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 4u(t)$$

$\to$ input



$s^2Y(s) + 2sY(s) + 4Y(s) = 4U(s)$

$Y(s)(s^2+2s+4) = 4U(s)$

$Y(s) = \frac{4}{s^2+2s+4}U(s)$.

Inverse Laplace Transform.

$s \to t$. $\mathcal{L}^{-1}\left\{\frac{4}{s^2+2s+4}\right\}$.

Example 1

$$\Sigma T = J\alpha = J\ddot{\theta}$$

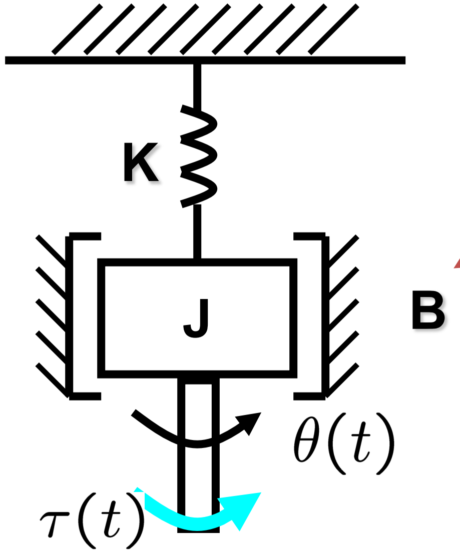
$$\tau(t) - K\theta(t) - B\cdot\dot{\theta}(t) = J\ddot{\theta}(t)$$

$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t) = \tau(t)$$

Zero initial conditions:

$$Js^2\Theta(s) + Bs\,\Theta(s) + K\,\Theta(s) = T(s)$$

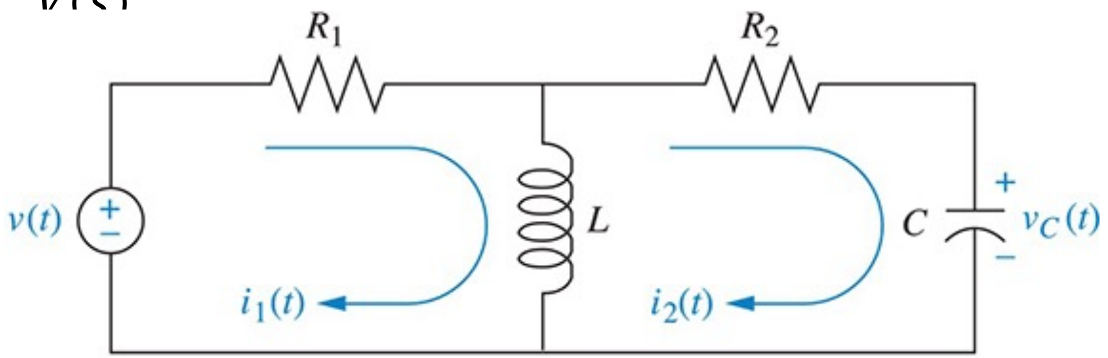
TF $$G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}.$$



friction between bob and air

Example 2

Given the network below, find the transfer function $\frac{I_2(s)}{V(s)}$.



$\frac{I_2(s)}{V(s)}$

TF $\Rightarrow$ $T(s) = \frac{I_2(s)}{V(s)}$

Impedance methods

Converting a TF to State Space

s domain

TF: Input-output model

$T(s) = \frac{\text{output}}{\text{Input}}$ s domain

$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ t. time domain

Assume the TF of a SISO system is as follows:

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{\boxed{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}} \text{ where } m < n$$

TF

Then its state-space model can be written below:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \text{where } A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}_{n \times n},$$

ss. A B C D. square

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}, C = [b_0 \quad b_1 \quad \cdots \quad b_m \quad 0]_{1 \times n}, D = [0]$$

$n > m$

Example

Input $\rightarrow \boxed{G(s)} \rightarrow$ output

Block diagram

TF $\Leftrightarrow$ SS.

$$G(s) = \frac{2s^2 + 5s + 3}{3s^3 + 7s^2 - 6s + 1}$$

$\uparrow$ 3

Please find its state-space model.

$$A_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ -\frac{1}{3} & 2 & -\frac{7}{3} \end{bmatrix}$$

$n = 3$

$$G(s) = \frac{\frac{2}{3}s^2 + \frac{5}{3}s + 1}{s^3 + \frac{7}{3}s^2 - 2s + \frac{1}{3}} \text{ (third-order system)}$$

$$B_{3\times 1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Its state-space model: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$$C_{1\times 3} = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & 2 & -\frac{7}{3} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}, D = [0]$$

$$D = [0].$$

Converting from State Space to TF

Assume the state-space model of a system is as follows:

$$\begin{cases} \dot{x} = Ax + Bu & \to \text{state equation} \\ y = Cx + Du & \to \text{output equation} \end{cases}$$

Take the Laplace Transform assuming zero initial conditions

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$X(s) = (sI - A)^{-1} B \cdot U(s).$$

Solving for X(s) in above equations

$X(s) = (sI - A)^{-1}BU(s)$ where I is the identity matrix

Substitute it to $y = Cx + Du$ ➔

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \boxed{C(sI - A)^{-1}B + D}$$

Example

state-space model:

$A_{3\times3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$ $C = [1 \ 0 \ 0]$ $D = [0]$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$ → state equation

$$y = [1 \quad 0 \quad 0] x + 0 \cdot u$$ → output equation

Please find its transfer function.

$$G(s) = C(sI - A)^{-1}B + D$$

$$= [1 \quad 0 \quad 0]\left(s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\right)^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$ ✓

$$= [1 \quad 0 \quad 0] \begin{bmatrix} s & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$

$$\boxed{= \frac{10s^2+30s+20}{s^3+3s^2+2s+1}}$$ TF.

Input → $\boxed{G(s) = \frac{10s^2+30s+20}{s^3+3s^2+2s+1}}$ → Output

The End!!